On the limiting behaviour of the spectra of a family of differential operators

2006 J. Phys. A: Math. Gen. 39 10477

(http://iopscience.iop.org/0305-4470/39/33/015)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 10477–10489 doi:10.1088/0305-4470/39/33/015

On the limiting behaviour of the spectra of a family of differential operators

Michael Solomyak

Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel

E-mail: michail.solomyak@weizmann.ac.il

Received 27 April 2006, in final form 3 July 2006
Published 2 August 2006
Online at stacks.iop.org/JPhysA/39/10477

Abstract

We study a family of self-adjoint partial differential operators $\mathbf{H}_\omega$, where ω is a large parameter. In the simplest case each operator acts in $L^2((a,b)\times\mathbb{R})$ as

$$\mathbf{H}_\omega = -\partial_x^2 + \omega\big(-\partial_y^2 + Q(y)\big),$$

under the boundary conditions of a certain type. We are interested in the behaviour of the eigenvalues $\lambda_n(\mathbf{H}_\omega)$ as $\omega\to\infty$. Let Λ_0 stand for the lowest eigenvalue of the Schrödinger operator $-\partial_y^2 + Q(y)$ in $L^2(\mathbb{R})$. Under some assumptions about the data we show that the numbers $\lambda_n(\mathbf{H}_\omega) - \omega\Lambda_0$ converge to the eigenvalues of the boundary value problem $-\psi'' = \lambda\psi$ on (a,b), under some boundary conditions induced by those for the original operators. Possible generalizations are also discussed.

PACS numbers: 02.30.Hq, 02.60.Lj, 03.65.−w
Mathematics Subject Classification: 34B45, 35P15

1. Introduction

In the study of operator families depending on a parameter, an important question concerns the limiting behaviour of their spectra. There are many different realizations of this general problem. One of them arises when each operator, say $\mathbf{H}_\omega$, is defined by a PDE acting on a set (a domain, or a manifold, etc) of some space dimension, while the limiting behaviour of the spectra $\sigma(\mathbf{H}_\omega)$ is determined by an operator $\mathbf{H}$ acting on a set of smaller dimension. The main goal in this type of problems is to construct this operator $\mathbf{H}$ and to establish precisely, how its spectrum determines the limiting behaviour of $\sigma(\mathbf{H}_\omega)$.

The problem investigated in the present paper is of this type. The operators studied act on the set $\Gamma\times\mathbb{R}$, where Γ is a metric graph. The general formulation of the problem requires some additional knowledge in graph theory and is given in section 2.3. Here we discuss a particular case, when Γ reduces to the single segment $[a,b]\subset\mathbb{R}$. Then $\mathbf{H}_\omega$ is the operator in $L^2((a,b)\times\mathbb{R})$ defined by the following differential expression and boundary conditions:

$$\mathbf{H}_\omega \Psi = -\Psi''_{xx}(x,y) + \omega(-\Psi''_{yy}(x,y) + Q(y)\Psi(x,y)), \qquad x \in (a,b), \quad y \in \mathbb{R}; \tag{1.1}$$

$$-\Psi'_x(a,y) = F_a(y)\Psi(a,y), \qquad \Psi'_x(b,y) = F_b(y)\Psi(a,y), \qquad y \in \mathbb{R}. \tag{1.2}$$

Here in the introduction we assume for simplicity that $Q(y) \to \infty$ as $|y| \to \infty$ and both F_a and F_b are bounded.

For large values of ω the operator $\mathbf{H}_\omega$ can be accurately defined via its quadratic form. Its spectrum is discrete, and we are interested in the behaviour of the eigenvalues $\lambda_n(\mathbf{H}_\omega)$ as $\omega \to \infty$. It is easy to see that $\lambda_n(\mathbf{H}_\omega) \to \infty$ for each n. In order to provide more essential information, let us denote by Λ_0 the lowest eigenvalue and by $U_0(y)$ the corresponding normalized eigenfunction of the Schrödinger operator $-u'' + Qu$ in $L^2(\mathbb{R})$. Then our main result, theorem 2.3, states that for each n

$$\lambda_n(\mathbf{H}_\omega) - \omega\Lambda_0 \to \lambda_n(\mathbf{H}), \qquad \omega \to \infty,$$

where $\mathbf{H}$ is the following operator in $L^2(a,b)$:

$$\mathbf{H}\psi = -\psi'', \qquad -\psi'(a) = K_a\psi(a), \qquad \psi'(b) = K_b\psi(b), \tag{1.3}$$

with the coefficients K_a, K_b given by

$$K_a = \int_{\mathbb{R}} F_a(y)U_0^2(y)\,\mathrm{d}y, \qquad K_b = \int_{\mathbb{R}} F_b(y)U_0^2(y)\,\mathrm{d}y. \tag{1.4}$$

The origin of the problem lies in the model of an irreversible quantum graph, suggested by Smilansky in [8]. Let us recall, see e.g. [5], that the term 'quantum graph' stands for the system consisting of a metric graph Γ and a differential operator on it, usually the Laplacian. In Smilansky model one studies the interaction between the quantum graph Γ and the harmonic oscillator, attached to Γ by means of some specific boundary conditions. If $\Gamma = [a,b]$, the model reduces to the problem described above, with

$$Q(y) = y^2, \qquad F_a(y) = F_b(y) = \alpha y. \tag{1.5}$$

Here $\alpha \geqslant 0$ is an additional parameter (the coupling constant) which characterizes the strength of interaction. For the case discussed we get $K_a = K_b = 0$, so that independently of the value of α the operator $\mathbf{H}$ in (1.3) is just the (minus) Neumann Laplacian on (a,b). The latter result (for arbitrary compact graphs) was obtained in [9] by using the decomposition into the Fourier–Hermite series. The result of the present paper can be considered as a generalization of theorem 2.2 in [9] to the case when a general Schrödinger operator, rather than just the harmonic oscillator, is attached to the quantum graph. The boundary conditions which describe the way of attaching also are of a wider class. The approach used in [9] does not apply, and the proof requires other technical tools. It is based upon lemma 3.1 of a rather general nature.

The main new feature of the result obtained is dependence of the boundary conditions for the limiting operator $\mathbf{H}$ on the functions involved in the boundary conditions for $\mathbf{H}_\omega$.

Let us briefly describe the structure of the paper. In section 2 we introduce the family $\mathbf{H}_\omega$ of the operators studied and formulate our main result, theorem 2.3. Its proof, given in section 4, is based on lemma 3.1, presented in section 3. In the final section 5 we complement and discuss theorem 2.3. In particular, we compare it with some results on the spectrum of the Laplacian in thin domains.

Here we introduce some necessary notation from the operator theory. Let $\mathbf{T}$ be a self-adjoint, bounded below operator in a separable Hilbert space $\mathcal{H}$. It is convenient to define such operator via its quadratic form $\mathbf{t}[f]$. The corresponding sesqui-linear form is denoted by $\mathbf{t}[f_1, f_2]$, so that $\mathbf{t}[f,f] = \mathbf{t}[f]$.

The spectrum and the essential spectrum of $\mathbf{T}$ are denoted by $\sigma(\mathbf{T})$ and by $\sigma_{\mathrm{ess}}(\mathbf{T})$, respectively. The spectrum below the point $\min \sigma_{\mathrm{ess}}(\mathbf{T})$ is either empty, or consists of a finite or countable set of eigenvalues $\{\lambda_n(\mathbf{T})\}$. It is common to enumerate them in the increasing order, according to their multiplicities. Given a real number s, we denote

$$\mathcal{N}(s; \mathbf{T}) = \dim E^{\mathbf{T}}(-\infty, s)\mathcal{H},$$

where $E^{\mathbf{T}}$ stands for the spectral measure of $\mathbf{T}$.

2. Description of the problem

2.1. Laplacian on a metric graph

Let Γ be a compact and connected metric graph, with the set of vertices $\mathcal{V}$ and the set of edges $\mathcal{E}$. For simplicity, we assume that the graph has no cycles. The length of an edge e is denoted by $|e|$, and we write

$$\varepsilon = \varepsilon(\Gamma) = \min_{e\in\mathcal{E}}|e|, \qquad M(\Gamma) = \#\mathcal{V}.$$

Let us identify each edge incident to v (notation $e \sim v$) with the segment $[0, |e|]$ in such a way that the point $t = 0$ corresponds to the vertex v. Then for any function ψ on the graph, which is smooth enough on each such edge, the expression

$$[\psi'](v) = \sum_{e\sim v}(\psi|_e)'(0)$$

is well defined. It plays an important role in the theory of the Laplacian on graphs.

The Sobolev space $H^1(\Gamma)$ is defined as the space of all continuous functions ψ on the graph Γ, such that the restriction of ψ to each edge e lies in $H^1(e)$. The canonical norm in $H^1(\Gamma)$ is given by

$$\|\psi\|^2_{H^1(\Gamma)} = \int_\Gamma (|\psi'|^2 + |\psi|^2)\,\mathrm{d}x.$$

With an arbitrary set of $M(\Gamma)$ real numbers, $\mathcal{K} = \{K_v\}_{v\in\mathcal{V}}$, we associate the quadratic form

$$\mathfrak{d}_{\mathcal{K}}[\psi] = \int_\Gamma |\psi'_x|^2\,\mathrm{d}x + \sum_{v\in\mathcal{V}} K_v|\psi(v)|^2, \qquad \psi \in H^1(\Gamma), \tag{2.1}$$

and the corresponding self-adjoint operator $-\Delta_{\mathcal{K}}$ in $L^2(\Gamma)$. Due to the compactness of Γ, the spectrum of $-\Delta_{\mathcal{K}}$ is discrete.

Independently of the choice of $\mathcal{K}$, the operator $\Delta_{\mathcal{K}}$ acts as $\Delta_{\mathcal{K}}\psi = \psi''$ on each edge. A function $\psi \in H^1(\Gamma)$ lies in its operator domain $\mathrm{Dom}(\Delta_{\mathcal{K}})$ if and only if it belongs to the Sobolev space $H^2(e)$ on each edge and satisfies the condition

$$[\psi'](v) = K_v\psi(v) \tag{2.2}$$

at each vertex. This description of $\mathrm{Dom}(\Delta_{\mathcal{K}})$ can be easily derived from the variational definition of the operator by using the Euler–Lagrange equations. The equality (2.2) with $K_v = 0$ is known as the Kirchhoff condition. The Laplacian under the Kirchhoff condition at each vertex $v \in \mathcal{V}$ is nothing but the Neumann Laplacian Δ_N.

2.2. The operator $\mathbf{A}_Q$

Another initial object is the Schrödinger operator $\mathbf{A}_Q u = -u'' + Qu$ in $L^2(\mathbb{R})$, where the potential Q is supposed measurable and non-negative. The operator $\mathbf{A}_Q$ can be accurately defined via its quadratic form

$$\mathbf{a}_Q[u] = \int_{\mathbb{R}} (|u'|^2 + Q(y)|u|^2)\,\mathrm{d}y, \qquad u \in \mathrm{Dom}(\mathbf{a}_Q) := H^1(\mathbb{R}) \cap L^2_Q(\mathbb{R}).$$

We formulate our conditions on the potential Q in an implicit form, in terms of the spectral properties of $\mathbf{A}_Q$.

Condition 2.1. *The potential Q is non-negative, the operator $\mathbf{A}_Q$ is positive definite in $L^2(\mathbb{R})$ and the bottom Λ_0 of its spectrum is an isolated eigenvalue.*

It is well known that this eigenvalue is simple and that the corresponding normalized eigenfunction $U_0(y)$ can be taken positive. We denote by Λ' the bottom of the rest of the spectrum:

$$\Lambda' = \min\{\lambda \in \sigma(\mathbf{A}_Q) : \lambda > \Lambda_0\}.$$

We do not impose any further conditions on the potential. Our assumptions can be summarized as follows:

$$\mathbf{A}_Q U_0 = \Lambda_0 U_0, \qquad U_0 > 0, \qquad \int_{\mathbb{R}} |U_0|^2 \, \mathrm{d}y = 1, \tag{2.3}$$

$$\mathbf{a}_Q[u] \geqslant \Lambda' \int_{\mathbb{R}} |u|^2 \, \mathrm{d}y \qquad \forall u \in \mathrm{Dom}(\mathbf{a}_Q) : \int_{\mathbb{R}} u U_0 \, \mathrm{d}y = 0, \qquad \Lambda' > \Lambda_0 > 0. \tag{2.4}$$

2.3. The operator $\mathbf{H}_\omega$

Now we are in a position to introduce the family of operators we study in this paper. Consider the set $\Gamma \times \mathbb{R}$, with the co-ordinates $x \in \Gamma$, $y \in \mathbb{R}$. In the usual way, the Hilbert space $\mathfrak{H} = L^2(\Gamma \times \mathbb{R})$ can be identified with the tensor product,

$$\mathfrak{H} = L^2(\Gamma \times \mathbb{R}) = L^2(\Gamma) \otimes L^2(\mathbb{R}). \tag{2.5}$$

We are interested in the operator $\mathbf{H}_\omega$ in $\mathfrak{H}$, defined by the differential expression

$$\mathbf{H}_\omega \Psi = -\Psi''_{xx} + \omega(-\Psi''_{yy} + Q(y)\Psi), \qquad x \notin \mathcal{V} \tag{2.6}$$

and the boundary (matching) conditions

$$[\Psi'_x](v, y) = F_v(y)\Psi(v, y), \qquad v \in \mathcal{V}, \quad y \in \mathbb{R}, \tag{2.7}$$

cf (1.1), (1.2). The potential Q is the same as in section 2.2 and $\omega > 0$ is a parameter; our assumptions about the functions F_v, which define the matching conditions at the vertices, will be specified later.

The most convenient way to rigorously define the operator $\mathbf{H}_\omega$ uses its quadratic form $\mathbf{h}_\omega$. Namely, we set

$$\mathbf{h}_\omega[\Psi] = \mathbf{h}_{\omega;0}[\Psi] + \mathbf{b}[\Psi], \tag{2.8}$$

where

$$\mathbf{h}_{\omega;0}[\Psi] = \int_{\Gamma \times \mathbb{R}} (|\Psi'_x|^2 + \omega(|\Psi'_y|^2 + Q(y)|\Psi|^2)) \, \mathrm{d}x \, \mathrm{d}y \tag{2.9}$$

and

$$\mathbf{b}[\Psi] = \sum_{v \in \mathcal{V}} \mathbf{b}_v[\Psi], \qquad \mathbf{b}_v[\Psi] = \int_R F_v(y)|\Psi(v, y)|^2 \, \mathrm{d}y. \tag{2.10}$$

The quadratic form $\mathbf{h}_{\omega;0}$ in (2.9) is defined on the natural domain $\mathfrak{D} := H^1(\Gamma) \otimes \mathrm{Dom}(\mathbf{a}_Q)$, which does not depend on $\omega > 0$. This quadratic form is non-negative and evidently closed in $\mathfrak{H}$. In part 1 of theorem 2.3, we will show that for the large values of ω the quadratic form $\mathbf{h}_\omega$

is bounded below and closed on $\mathfrak{D}$. Therefore, it defines a self-adjoint operator in $\mathfrak{H}$ which by definition is taken as $\mathbf{H}_\omega$.

The operator $\mathbf{H}_{\omega;0}$, associated with the quadratic form (2.9), admits separation of variables: with respect to the tensor realization (2.5) of $\mathfrak{H}$, we have

$$\mathbf{H}_{\omega;0} = -\Delta_N \otimes \mathbf{I} + \omega \mathbf{I} \otimes \mathbf{A}_Q,$$

where Δ_N is the Neumann Laplacian on Γ; see a description of this operator at the end of section 2.1. It follows that the spectrum of $\mathbf{H}_{\omega,0}$ is the superposition of the sets $\lambda_n(-\Delta_N) + \omega\sigma(\mathbf{A}_Q)$. In particular, the spectrum of $\mathbf{H}_{\omega;0}$ below $\omega\Lambda'$ is finite, it consists of the eigenvalues

$$\lambda_n(\mathbf{H}_{\omega;0}) = \lambda_n(-\Delta_N) + \omega\Lambda_0, \qquad n \leqslant \mathcal{N}(\omega(\Lambda' - \Lambda_0); -\Delta_N).$$

Hence,

$$\mathcal{N}(s; \mathbf{H}_{\omega;0}) = \mathcal{N}(s; \mathbf{A}_Q), \qquad \forall s \leqslant \omega(\Lambda' - \Lambda_0).$$

The number $\mathcal{N}(\omega(\Lambda' - \Lambda_0); -\Delta_N)$ indefinitely grows together with ω, and it is clear that

$$\lim_{\omega\to\infty} (\lambda_n(\mathbf{H}_{\omega;0}) - \omega\Lambda_0) = \lambda_n(-\Delta_N), \qquad \forall n \geqslant 0. \tag{2.11}$$

Our goal is to study the behaviour of the eigenvalues $\lambda_n(\mathbf{H}_\omega)$ as $\omega \to \infty$. We shall show that a formula similar to (2.11) takes place, but with an important difference: the limiting eigenvalues on the right-hand side correspond not to the Neumann Laplacian Δ_N, but to the operator $\Delta_{\mathcal{K}}$ where

$$\mathcal{K} = \{K_v\}, \qquad K_v = \int_{\mathbb{R}} F_v(y) U_0^2(y)\,\mathrm{d}y, \qquad \forall\, v \in \mathcal{V}. \tag{2.12}$$

The conditions on the functions F_v should guarantee finiteness of K_v.

Lemma 2.2. *Let Q be as above, and let the functions F_v be measurable and satisfy the inequality*

$$|F_v(y)| \leqslant C(1 + \sqrt{Q(y)}), \qquad C > 0, \quad \forall\, v \in \mathcal{V}. \tag{2.13}$$

Then the numbers K_v given by (2.12) meet the estimate

$$|K_v| \leqslant C(1 + \sqrt{\Lambda_0}).$$

Proof. We have

$$\int_{\mathbb{R}} Q(y)|U_0(y)|^2\,\mathrm{d}y \leqslant \mathbf{a}_Q[U_0] = \Lambda_0, \qquad \int_{\mathbb{R}} |U_0(y)|^2\,\mathrm{d}y = 1.$$

Hence, by the Cauchy–Schwartz inequality,

$$\int_{\mathbb{R}} (1 + \sqrt{Q(y)})|U_0(y)|^2\,\mathrm{d}y \leqslant 1 + \sqrt{\Lambda_0}.$$

The desired result immediately follows. □

2.4. *Main result*

Theorem 2.3. *Let condition 2.1 be satisfied, the functions F_v be measurable and real-valued and the inequalities (2.13) be fulfilled. Then*

(1) For ω large enough the quadratic form $\mathbf{h}_\omega$ given by (2.8) is bounded below and closed on $\mathfrak{D}$.

Let $\mathbf{H}_\omega$ stand for the corresponding self-adjoint operator in $\mathfrak{H}$.

(2) There exists a function $s(\omega)$, such that $s(\omega)\to\infty$ as $\omega\to\infty$ and

$$\mathcal{N}(s(\omega);\mathbf{H}_\omega)<\infty \quad \text{for each } \omega<\infty, \qquad \mathcal{N}(s(\omega);\mathbf{H}_\omega)\to\infty \quad \text{as } \omega\to\infty. \tag{2.14}$$

(3) For any $n\in\mathbb{N}$ we have

$$\lambda_n(\mathbf{H}_\omega)-\omega\Lambda_0\to\lambda_n(-\Delta_{\mathcal{K}}) \quad \text{as } \omega\to\infty, \tag{2.15}$$

where the family $\mathcal{K}$ is given by (2.12).

We reduce the theorem to a technical lemma which is stated and proved in the following section.

3. The main lemma

The lemma below describes the central construction used for the proof of theorem 2.3. The assumptions about the objects involved reflect the main features of the problem we are dealing with.

3.1. The quadratic form $\mathbf{g}_\omega[\Psi]$

Let $\mathcal{H}$ be a separable Hilbert space and $\mathbf{g}_\omega[\Psi]$ be a positive definite, closed quadratic form in $\mathcal{H}$, depending on a parameter $\omega>0$. We suppose that the domain $\mathfrak{V}=\mathrm{Dom}[\mathbf{g}_\omega]$ is dense in $\mathcal{H}$ and does not depend on ω. Below we list our assumptions about the family of the quadratic forms $\mathbf{g}_\omega$ and the corresponding self-adjoint operators $\mathbf{G}_\omega$ in $\mathcal{H}$.

(g1) *For any $\omega>0$ the quadratic form $\mathbf{g}_\omega$ is diagonalized by an orthogonal decomposition*

$$\mathcal{H}=\mathcal{H}^\circ\oplus\widetilde{\mathcal{H}}, \qquad \dim\mathcal{H}^\circ=\infty, \tag{3.1}$$

which does not depend on ω.

In the following the elements from $\mathcal{H}^\circ$ and $\widetilde{\mathcal{H}}$ are standardly denoted by Ψ° and $\widetilde{\Psi}$ respectively, and $\mathbf{P}^\circ$, $\widetilde{\mathbf{P}}$ stand for the orthogonal projections in $\mathcal{H}$ onto these subspaces.

(g2) *The part of $\mathbf{g}_\omega$ in the subspace $\mathcal{H}^\circ$ is independent of ω:*

$$\mathbf{g}_\omega[\Psi^\circ]=:\mathbf{g}^\circ[\Psi^\circ], \qquad \forall\,\Psi^\circ\in\mathfrak{V}\cap\mathcal{H}^\circ,$$

and the self-adjoint operator $\mathbf{G}^\circ$ in $\mathcal{H}^\circ$, associated with $\mathbf{g}^\circ$, has discrete spectrum.

(g3) *The lower bound of the quadratic form $\mathbf{g}_\omega$ on $\widetilde{\mathcal{H}}$ indefinitely grows together with ω. More exactly, there exists a constant $\nu>0$, such that*

$$\mathbf{g}_\omega[\widetilde{\Psi}]\geqslant\nu\omega\|\widetilde{\Psi}\|^2_{\mathcal{H}}, \qquad \forall\,\widetilde{\Psi}\in\mathfrak{V}\cap\widetilde{\mathcal{H}}.$$

Let $\widetilde{\mathbf{G}}_\omega$ stand for the self-adjoint operator in $\widetilde{\mathcal{H}}$, generated by the quadratic form $\mathbf{g}_\omega$ restricted to $\widetilde{\mathcal{H}}$. Our assumptions imply that

$$\sigma(\mathbf{G}_\omega)=\sigma(\mathbf{G}^\circ)\cup\sigma(\widetilde{\mathbf{G}}_\omega).$$

It follows that the spectra of $\mathbf{G}^\circ$ and $\mathbf{G}_\omega$, lying below the point $\nu\omega$, coincide. In particular,

$$\mathcal{N}(s;\mathbf{G}_\omega)=\mathcal{N}(s;\mathbf{G}^\circ), \qquad \forall\,s\leqslant\nu\omega. \tag{3.2}$$

3.2. The quadratic form $\mathbf{m}[\Psi]$

Let $\mathbf{m}[\Psi]$ be another real-valued quadratic form (not depending on ω) in $\mathcal{H}$ defined on the same domain $\mathfrak{V}$. Our first assumption about $\mathbf{m}$ is this:

(**m1**) *The quadratic form* **m** *vanishes on* $\mathcal{H}^\circ$*:*

$$\mathbf{m}[\Psi^\circ] = 0, \qquad \forall\, \Psi^\circ \in \mathfrak{V} \cap \mathcal{H}^\circ.$$

The second assumption is formulated in terms of a non-negative majorant of **m**, i.e. a non-negative quadratic form $\widehat{\mathbf{m}}$ such that

$$|\mathbf{m}[\Psi]| \leqslant \widehat{\mathbf{m}}[\Psi], \qquad \forall\, \Psi \in \mathfrak{V}. \tag{3.3}$$

(**m2**) *There are a constant* $C_1 \geqslant 0$ *and a function* $\varepsilon(\omega)$*, vanishing as* $\omega \to \infty$*, such that*

$$\widehat{\mathbf{m}}[\Psi^\circ] \leqslant C_1 \mathbf{g}^\circ[\Psi^\circ], \qquad \forall\, \Psi^\circ \in \mathfrak{V} \cap \mathcal{H}^\circ, \tag{3.4}$$

$$\widehat{\mathbf{m}}[\widetilde{\Psi}] \leqslant \varepsilon(\omega) \mathbf{g}_\omega[\widetilde{\Psi}], \qquad \forall\, \widetilde{\Psi} \in \mathfrak{V} \cap \widetilde{\mathcal{H}}. \tag{3.5}$$

We do not assume that the decomposition (3.1) diagonalizes either of the quadratic forms **m**, $\widehat{\mathbf{m}}$.

Lemma 3.1. *Let the assumptions* (**g1**)–(**g3**) *and* (**m1**), (**m2**) *be satisfied. Then*

(1) For ω *large enough the quadratic form*

$$\mathbf{g}^{\mathbf{m}}_\omega[\Psi] := \mathbf{g}_\omega[\Psi] + \mathbf{m}[\Psi], \qquad \Psi \in \mathfrak{V},$$

is bounded below and closed in $\mathcal{H}$.
Let $\mathbf{G}^{\mathbf{m}}_\omega$ *stand for the corresponding self-adjoint operator.*

(2) There exists a function $s_1(\omega)$*, such that* $s_1(\omega) \to \infty$ *as* $\omega \to \infty$ *and*

$$\begin{aligned} &\mathcal{N}\big(s_1(\omega); \mathbf{G}^{\mathbf{m}}_\omega\big) < \infty \quad \text{for each } \omega < \infty, \\ &\mathcal{N}\big(s_1(\omega); \mathbf{G}^{\mathbf{m}}_\omega\big) \to \infty \quad \text{as } \omega \to \infty. \end{aligned} \tag{3.6}$$

(3) For any $n \in \mathbb{N}$ *we have*

$$\lambda_n\big(\mathbf{G}^{\mathbf{m}}_\omega\big) \to \lambda_n(\mathbf{G}^\circ) \quad \text{as } \omega \to \infty. \tag{3.7}$$

Proof. (1) Let $\Psi = \Psi^\circ + \widetilde{\Psi}$, then by the assumption (**m1**)

$$\mathbf{g}^{\mathbf{m}}_\omega[\Psi] = \mathbf{g}_\omega[\Psi] + \mathbf{m}[\widetilde{\Psi}] + 2\,\mathrm{Re}\,\mathbf{m}[\Psi^\circ, \widetilde{\Psi}]. \tag{3.8}$$

It follows from the 'polarization identity'

$$4\mathbf{m}[\Psi^\circ, \widetilde{\Psi}] = \sum_{k=0}^{3} \mathrm{i}^k \mathbf{m}[\Psi^\circ + \mathrm{i}^k \widetilde{\Psi}]$$

and the property of $\mathbf{m}[\Psi]$ to be real-valued that

$$4\,\mathrm{Re}\,\mathbf{m}[\Psi^\circ, \widetilde{\Psi}] = \mathbf{m}[\Psi^\circ + \widetilde{\Psi}] - \mathbf{m}[\Psi^\circ - \widetilde{\Psi}].$$

Hence, by (3.3)

$$2|\mathrm{Re}\,\mathbf{m}[\Psi^\circ, \widetilde{\Psi}]| \leqslant \widehat{\mathbf{m}}[\Psi^\circ] + \widehat{\mathbf{m}}[\widetilde{\Psi}]. \tag{3.9}$$

Choosing an arbitrary $\delta > 0$ and replacing in (3.9) Ψ° by $\delta^{1/2}\Psi^\circ$ and $\widetilde{\Psi}$ by $\delta^{-1/2}\widetilde{\Psi}$, we conclude from (3.8) that

$$\big|\mathbf{g}^{\mathbf{m}}_\omega[\Psi] - \mathbf{g}_\omega[\Psi]\big| \leqslant \widehat{\mathbf{m}}[\widetilde{\Psi}] + \delta\widehat{\mathbf{m}}[\Psi^\circ] + \delta^{-1}\widehat{\mathbf{m}}[\widetilde{\Psi}].$$

Let now ω be large enough, so that $\varepsilon(\omega) \leqslant 1$. Taking $\delta = \sqrt{\varepsilon(\omega)}$, we find that

$$\begin{aligned} \big|\mathbf{g}^{\mathbf{m}}_\omega[\Psi] - \mathbf{g}_\omega[\Psi]\big| &\leqslant \sqrt{\varepsilon(\omega)}\,\big((2 + C_1)\mathbf{g}^{\mathbf{m}}_\omega[\Psi^\circ] + \mathbf{g}_\omega[\widetilde{\Psi}]\big) \\ &\leqslant (2 + C_1)\sqrt{\varepsilon(\omega)}\mathbf{g}_\omega[\Psi]. \end{aligned} \tag{3.10}$$

If $\mu(\omega) := (2+C_1)\sqrt{\varepsilon(\omega)} < 1$, this inequality yields (see, e.g., lemma 1.1 in [1]) that together with $\mathbf{g}_\omega$, the quadratic form $\mathbf{g}^{\mathbf{m}}_\omega$ is bounded below and closed on $\mathfrak{d}$. This completes the proof of (1). □

(2) From (3.10) we conclude that

$$(1-\mu(\omega))\mathbf{G}_\omega \leqslant \mathbf{G}^{\mathbf{m}}_\omega \leqslant (1+\mu(\omega))\mathbf{G}_\omega.$$

Therefore, for any $s > 0$

$$\mathcal{N}\left(\frac{s}{1+\mu(\omega)}; \mathbf{G}_\omega\right) \leqslant \mathcal{N}\big(s; \mathbf{G}^{\mathbf{m}}_\omega\big) \leqslant \mathcal{N}\left(\frac{s}{1-\mu(\omega)}; \mathbf{G}_\omega\right). \tag{3.11}$$

If $s < \nu\omega(1-\mu(\omega))$, then by (3.2) the operator $\mathbf{G}_\omega$ in the latter inequality can be replaced by $\mathbf{G}^\circ$. Take, for instance,

$$s_1(\omega) = \frac{\nu\omega}{2}(1-\mu(\omega)).$$

Then the quantity $N\big(s_1(\omega); \mathbf{G}^{\mathbf{m}}_\omega\big)$ is finite and tends to infinity together with ω (since $\mu(\omega) \to 0$). This proves (2).

From (3.11) the statement (3) of lemma follows immediately. □

4. Proof of theorem 2.4.3

We rely upon lemma 3.1. So, we have to define the appropriate quadratic forms $\mathbf{g}_\omega$, $\mathbf{m}$ and $\widehat{\mathbf{m}}$ and then to check the conditions (**g1**)–(**g3**) and (**m1**), (**m2**).

Our Hilbert space is $\mathcal{H} = L^2(\Gamma \times \mathbb{R})$, and all the quadratic forms involved are defined on the domain $\mathfrak{D} = H^1(\Gamma) \otimes \mathrm{Dom}(\mathbf{a}_Q)$. As in section 2.2, $U_0(y)$ stands for the leading (positive), normalized eigenfunction of the operator $\mathbf{A}_Q$, see (2.3). We define the subspace $\mathcal{H}^\circ$ as

$$\mathcal{H}^\circ = \{\Psi(x,y) = \psi(x)U_0(y) : \psi \in L^2(\Gamma)\} \tag{4.1}$$

and the subspace $\widetilde{\mathcal{H}}$ as its orthogonal complement in $\mathfrak{H}$. The orthogonal projection onto $\mathcal{H}^\circ$ is given by

$$(\mathbf{P}^\circ\Psi)(x,y) = \psi(x)U_0(y), \qquad \psi(x) = \int_{\mathbb{R}} \Psi(x,y)U_0(y)\,\mathrm{d}y.$$

We define the quadratic form $\mathbf{g}_\omega$ as

$$\mathbf{g}_\omega[\Psi] = \mathbf{h}_{\omega;0}[\Psi] - (\omega\Lambda_0 - R)\|\Psi\|^2 + \mathbf{b}[\mathbf{P}^\circ\Psi]. \tag{4.2}$$

Here $\mathbf{h}_{\omega;0}$ and $\mathbf{b}$ are the quadratic forms (2.9) and (2.10), R is a non-negative constant to be specified later and $\|\Psi\| = \|\Psi\|_{\mathfrak{H}}$. Since U_0 is an eigenfunction of $\mathbf{A}_Q$, the decomposition $\mathcal{H} = \mathcal{H}^\circ \oplus \widetilde{\mathcal{H}}$ diagonalizes the quadratic form $\mathbf{g}_\omega$.

In particular,

$$\mathbf{g}_\omega[\Psi^\circ] = \int_\Gamma (|\psi'_x|^2 + R|\psi|^2)\,\mathrm{d}x + \sum_{v\in\mathcal{V}} K_v|\psi(v)|^2 = \mathfrak{d}_{\mathcal{K}}[\psi] + R\|\psi\|^2_{L^2(\Gamma)} \tag{4.3}$$

(where $\mathfrak{d}_{\mathcal{K}}$ is the quadratic form defined in (2.1), with the coefficients K_v from (2.12)) and

$$\mathbf{g}_\omega[\widetilde{\Psi}] = \int_{\Gamma\times\mathbb{R}} ((|\widetilde{\Psi}'_x|^2 + R|\widetilde{\Psi}|^2) + \omega(|\widetilde{\Psi}'_y|^2 + (Q(y) - \Lambda_0)|\widetilde{\Psi}|^2))\,\mathrm{d}x\,\mathrm{d}y. \tag{4.4}$$

The choice of R in (4.2) has to guarantee positive definiteness of the quadratic form in (4.3). Due to the continuous imbedding of the Sobolev space $H^1(\Gamma)$ in $C(\Gamma)$, such choice is always possible. It depends on the functions F_v involved in (2.10).

For any $u \in \mathrm{Dom}(\mathbf{a}_Q)$, orthogonal to U_0, the inequality (2.4) is satisfied. It follows that for any $\widetilde{\Psi} \in \mathfrak{D} \cap \widetilde{\mathcal{H}}$ we have

$$\mathbf{g}_\omega[\widetilde{\Psi}] \geqslant (\Lambda' - \Lambda_0)\|\widetilde{\Psi}\|^2,$$

and by (4.2)

$$\mathbf{g}_\omega[\Psi] = \mathbf{g}_\omega[\Psi^\circ] + \mathbf{g}_\omega[\widetilde{\Psi}] \geqslant \mathfrak{d}_{\mathcal{K}}[\Psi^\circ] + \omega(\Lambda' - \Lambda_0)\|\widetilde{\Psi}\|^2. \tag{4.5}$$

This shows that the quadratic form $\mathbf{g}_\omega$ is positive definite. It is evidently closed. Hence, the condition **(g1)** is satisfied. The equality (4.3) implies that the condition (**g2**) is satisfied too. The operator $\mathbf{G}^\circ$ acts as

$$(\mathbf{G}^\circ\Psi^\circ)(x, y) = (-\Delta_{\mathcal{K}}\psi(x) + R\psi(x))U_0(y).$$

By (4.5), the condition (**g3**) is fulfilled with $\nu = \Lambda' - \Lambda_0$.

Now we introduce the quadratic form **m**:

$$\mathbf{m}[\Psi] = \mathbf{b}[\Psi] - \mathbf{b}[\mathbf{P}^\circ\Psi]. \tag{4.6}$$

The condition (**m1**) is satisfied automatically. Note that (4.2) and (4.6) yield

$$\mathbf{g}_\omega^{\mathbf{m}}[\Psi] = \mathbf{g}_\omega[\Psi] + \mathbf{m}[\Psi] = \mathbf{h}_\omega[\Psi] - (\omega\Lambda_0 - R)\|\Psi\|^2$$

and therefore,

$$\mathbf{G}_\omega^{\mathbf{m}} = \mathbf{H}_\omega - (\omega\Lambda_0 - R)\mathbf{I}. \tag{4.7}$$

Take

$$\widehat{\mathbf{m}}[\Psi] = C_2 \sum_{v\in\mathcal{V}(\Gamma)} \int_{\mathbb{R}} (1 + \sqrt{Q(y)})|\Psi(v, y)|^2\,\mathrm{d}y.$$

Due to the assumption (2.13), we have $|\mathbf{b}[\Psi]| \leqslant \widehat{\mathbf{m}}[\Psi]$ if $C_2 \geqslant C$. Further, let $\Psi \in \mathfrak{D}$ and $\psi(x) = \int_{\mathbb{R}} \Psi(x, y)U_0(y)\,\mathrm{d}y$. Then for any $x \in \Gamma$

$$|\psi(x)|^2 \leqslant \int_{\mathbb{R}} |\Psi(x, y)|^2\,\mathrm{d}y \leqslant \int_{\mathbb{R}} (1 + \sqrt{Q(y)})|\Psi(x, y)|^2\,\mathrm{d}y.$$

Since

$$\mathbf{b}[\mathbf{P}^\circ\Psi] = \sum_{v\in\mathcal{V}(\Gamma)} K_v|\psi(v)|^2$$

and all the K_v are estimated by lemma 2.2; the inequality (3.3), with an appropriate choice of C_2, is satisfied.

It remains to verify conditions (3.4) and (3.5). The first is easy. Indeed, if $\Psi = \Psi^\circ$ is as in (4.1), then

$$\widehat{\mathbf{m}}[\Psi^\circ] = C_2 \sum_{v\in\mathcal{V}} |\psi(v)|^2 \int_{\mathbb{R}} (1 + \sqrt{Q(y)})|U_0(y)|^2\,\mathrm{d}y.$$

Again, the integrals are finite and the factors $|\psi(v)|^2$ are controlled by $\|\psi\|_{H^1(\Gamma)}$, and hence, by the quadratic form $\mathbf{g}^\circ[\Psi^\circ]$.

4.1. Verifying (3.5)

This requires some preliminary work. We shall rely upon the following elementary inequality which was exploited earlier in [10] where it was presented without proof. For the reader's convenience, we prove it here. Another proof can be found in [9].

Lemma 4.1. *Let $0 < L < \infty$. For any function $w(x)$ from the Sobolev space $H^1(0, L)$ and any number $\gamma > 0$ the inequality is satisfied:*

$$\gamma|w(0)|^2 \leqslant \coth(\gamma L) \int_0^L (|w_x'|^2 + \gamma^2|w|^2)\,dx. \tag{4.8}$$

Proof. By scaling, we reduce the problem to its particular case $\gamma = 1$, then the integral in (4.8) is nothing but the standard metric form $\|w\|^2_{H^1}$ of the space $H^1(0, L)$. Let $(w_1, w_2)_{H^1}$ stand for the corresponding scalar product.

Let $w_0(x) = \cosh(L - x)$, then for any $w_1 \in H^1(0, L)$ we get by integrating by parts

$$(w_1, w_0)_{H^1} = \int_0^L (w_1' w_0' + w_1 w_0)\,dx = w_1(0)\sinh L.$$

Hence,

$$(w_1, w_0)_{H^1} = 0 \quad \Longleftrightarrow \quad w_1(0) = 0.$$

Any function $w \in H^1(0, L)$ can be represented as $w = Cw_0 + w_1$, where $(w_1, w_0)_{H^1} = 0$. Then $w(0) = Cw_0(0) = C\cosh L$ and

$$\|w\|^2_{H^1} \geqslant |C|^2\|w_0\|^2_{H^1} = |C|^2\sinh(2L)/2 = |w(0)|^2\tanh L,$$

whence the result. □

Recall that $\coth s$, $s > 0$ is a decreasing function. By applying (4.8) to each vertex $v \in \mathcal{V}$ and roughening the estimate, we conclude that

$$\gamma|w(v)|^2 \leqslant \coth(\gamma\varepsilon) \int_\Gamma (|w'|^2 + \gamma^2|w|^2)\,dx, \qquad \forall\, v \in \mathcal{V}. \tag{4.9}$$

By (4.9), we have for all $v \in \mathcal{V}$ and for a.a. $y \in \mathbb{R}$:

$$\sqrt{\omega Q(y) + R}\,|\widetilde{\Psi}(v, y)|^2 \leqslant \coth\left(\varepsilon\sqrt{\omega Q(y) + R}\right) \times \int_\Gamma (|\widetilde{\Psi}_x'(x, y)|^2 + (\omega Q(y) + R)|\widetilde{\Psi}(x, y)|^2)\,dx,$$

whence

$$\sqrt{\omega Q(y)}\,|\widetilde{\Psi}(v, y)|^2 \leqslant \coth(\varepsilon\sqrt{R}) \int_\Gamma (|\widetilde{\Psi}_x'(x, y)|^2 + (\omega Q(y) + R)|\widetilde{\Psi}(x, y)|^2)\,dx.$$

Integrating this inequality over $y \in \mathbb{R}$ and increasing the right-hand side, we obtain

$$\sqrt{\omega}\int_{\mathbb{R}} \sqrt{Q(y)}\,|\widetilde{\Psi}(v, y)|^2\,dy \leqslant \coth(\varepsilon\sqrt{R}) \times \int_{\Gamma\times\mathbb{R}} (|\widetilde{\Psi}_x'|^2 + R|\widetilde{\Psi}|^2 + \omega(|\widetilde{\Psi}_y'|^2 + Q(y)|\widetilde{\Psi}|^2))\,dx\,dy. \tag{4.10}$$

Now we use the inequality (2.4) in an equivalent form,

$$\int_{\mathbb{R}} (|u_y'|^2 + Q(y)|u|^2)\,dy \leqslant \frac{\Lambda'}{\Lambda' - \Lambda_0}\int_{\mathbb{R}} (|u_y'|^2 + (Q(y) - \Lambda_0)|u|^2)\,dy.$$

Applying it to the function $u(\cdot) = \widetilde{\Psi}(x, \cdot)$, we conclude from (4.10) that

$$\int_{\mathbb{R}} \sqrt{\omega Q(y)} |\widetilde{\Psi}(v, y)|^2 \, \mathrm{d}y \leqslant \frac{\Lambda'}{\Lambda' - \Lambda_0} \coth(\varepsilon\sqrt{R}) \mathbf{g}_\omega[\widetilde{\Psi}]. \tag{4.11}$$

We also have to estimate the integral $\int_{\mathbb{R}} |\widetilde{\Psi}(v, y)|^2 \, \mathrm{d}y$. To this end, we first of all derive from (2.4) and (4.4) that

$$\int_{\Gamma \times \mathbb{R}} (|\widetilde{\Psi}'_x|^2 + \omega(\Lambda' - \Lambda_0)|\widetilde{\Psi}|^2) \, \mathrm{d}x \, \mathrm{d}y \leqslant \mathbf{g}_\omega[\widetilde{\Psi}]. \tag{4.12}$$

It follows from here and (4.9) that

$$\sqrt{\omega(\Lambda' - \Lambda_0)} \int_{\mathbb{R}} |\widetilde{\Psi}(v, y)|^2 \, \mathrm{d}y \leqslant \mathbf{g}_\omega[\widetilde{\Psi}].$$

The desired inequality (3.5), with

$$\varepsilon(\omega) = M(\Gamma)\omega^{-1/2} \left(\frac{\Lambda'}{\Lambda' - \Lambda_0} \coth(\varepsilon\sqrt{R}) + \frac{1}{\sqrt{\Lambda' - \Lambda_0}} \right),$$

is a direct consequence of (4.11) and (4.12).

4.2. End of proof

It follows from (4.7) that the statement (1) in theorem 2.3 is an immediate consequence of the statement (1) in lemma 3.1. The same equality (4.7) yields that

$$\mathcal{N}(s; \mathbf{H}_\omega) = \mathcal{N}\big(s - (\omega\Lambda_0 - R); \mathbf{G}^{\mathbf{m}}_\omega\big).$$

Hence, (2.14) follows from (3.6), with

$$s(\omega) = s_1(\omega) + (\omega\Lambda_0 - R).$$

Finally, (2.15) follows from (3.7).

5. Complements and general discussion

5.1. Some additional results

(1) Suppose that the functions F_v in (2.10) are such that

$$\int_{\mathbb{R}} F_v(y) U_0^2(y) \, \mathrm{d}y = 0, \qquad \forall \, v \in \mathcal{V}. \tag{5.1}$$

Then by (2.13) we get $K_v = 0$, so that the limiting behaviour of the eigenvalues $\lambda_n(\mathbf{H}_\omega)$ is determined by the Neumann Laplacian on Γ: relation (2.15) takes the form

$$\lambda_n(\mathbf{H}_\omega) - \omega\Lambda_0 \to \lambda_n(-\Delta_{\mathcal{N}}) \qquad \textit{as } \omega \to \infty. \tag{5.2}$$

Moreover, it is clear that condition (5.1) is not only sufficient, but also necessary for (5.2) to be satisfied.

In the paper [9] a special case of the operator $\mathbf{H}_\omega$ was considered, with $Q(y) = y^2$ and $F_v(y) = \alpha y$ for all $v \in \mathcal{V}$. Here $\alpha > 0$ is the coupling constant, cf equation (1.5) in the introduction. To indicate the dependence of α in our notation, below we denote the operator by $\mathbf{H}_{\omega,\alpha}$.

In the case discussed condition (5.1) is evidently fulfilled, so that theorem 2.2 in [9] follows from our theorem 2.3. The proof in [9] is easier than here, it uses expansions into the Fourier–Hermite series. This approach allows one to carry out a detailed spectral analysis of the operators $\mathbf{H}_{\omega,\alpha}$ for ω fixed and an arbitrary $\alpha > 0$. In this connection, see the paper [11] which presents such analysis for the case when $Q(y) = y^2$, $F_{v_0}(y) = \alpha y$ for a selected vertex v, and $F_v(y) \equiv 0$ for all $v \neq v_0$.

(2) In the description of the problem as in section 2.3 it is not necessary to take the Neumann Laplacian as the initial operator on Γ. It is possible to replace it by the operator $\Delta_{\mathcal{K}_0}$, with an arbitrary family $\mathcal{K}_0 = \{K_{v,0}\}_{v\in\mathcal{V}}$ of real numbers. In other words, the quadratic form $\mathbf{h}_{\omega;0}$ in (2.9) can be replaced by

$$\mathbf{h}_{\omega;0,\mathcal{K}_0}[\Psi] = \mathbf{h}_{\omega;0}[\Psi] + \sum_{v\in\mathcal{V}} K_{v,0} \int_{\mathbb{R}} |\Psi(v, y)|^2 \, \mathrm{d}y.$$

But this is only a formal generalization of the original problem: the same can be achieved by replacing the functions $F_v(y)$ by $F_v(y) + K_{v,0}$.

(3) The suggested scheme also applies when the operator attached to the quantum graph acts in L^2 on the half line, or on a finite interval. In particular, instead of the operator $\mathbf{A}_Q$ as in section 2.2 one can consider the Sturm–Liuville operator on the interval $(-l, l)$, with the Dirichlet boundary conditions at $y = \pm l$. This means that the differential expression in (2.6) is considered for $x \in \Gamma$, $y \in (-l, l)$, under the conditions (2.7) and $\Psi(v, \pm l) = 0$. As before, the quadratic form $\mathbf{h}_\omega$ is given by (2.8), with the obvious changes in (2.9) and (2.10).

An analogue of theorem 2.3 remains valid for this case without any changes. In particular, if $Q(y) \equiv 0$ (the infinite potential barrier at $y = \pm l$), then $\Lambda_0 = \frac{\pi^2}{4l^2}$ and $U_0(y) = l^{-1/2} \cos \frac{\pi x}{2l}$. This allows one to write the expression for the coefficients K_v in an explicit form.

(4) We come to another generalization of theorem 2.3 by replacing in (2.6) the term $-\Psi''_{xx}$ by $-\Psi''_{xx} + V(x)$, with a real-valued, bounded potential V. Clearly, the operator $-\Delta_{\mathcal{K}}$ on the right-hand side of (2.15) in theorem 2.3 has to be replaced by $-\Delta_{\mathcal{K}} + V$.

In the latter result the compactness assumption for Γ can be removed, it is enough to require that $M(\Gamma) < \infty$ and $V(x)$ is bounded on compact subsets of Γ and tends to infinity as $|x| \to \infty$ along the infinite edges. Then the spectrum of $-\Delta + V$ is discrete and the result remains valid. The proof consists in applying lemma 3.1, and only some minor changes in the argument of section 4 are necessary.

5.2. *On the spectrum of the Laplacian in thin domains*

There is some resemblance between theorem 2.3 and the results on the spectrum of the Dirichlet Laplacian Δ_D in a thin neighbourhood of a smooth simple planar curve $\mathcal{L}$. Let us assume for simplicity that $\mathcal{L}$ is closed. Then for any $d > 0$ the set

$$\Omega_d = \{(x, y) \in \mathbb{R}^2 : \mathrm{dist}((\mathrm{x}, \mathrm{y}), \mathcal{L}) < \mathrm{d}\}$$

is a bounded planar domain, so that the spectrum of $-\Delta_D$ in Ω_d is discrete. Let $\lambda_n(d)$ be its nth eigenvalue. When $d \to 0$, they escape to infinity in such a way that under some mild smoothness conditions on the curvature of $\mathcal{L}$ one has

$$\lambda_n(d) - \frac{\pi^2}{4d^2} \to \lambda_n(\mathbf{H}), \qquad \mathbf{H} = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{\gamma^2(s)}{4}, \tag{5.3}$$

where s is the arc length coordinate on $\mathcal{L}$ and $\gamma(s)$ is the curvature.

This result goes back to the paper [3]. In fact, the case of a non-compact curve was analysed there, when the spectra of the operators involved are not discrete, so that some changes in the formulation are necessary. The proof of (5.3) is actually the same as for the corresponding statement in [3], so that the result should be considered as known. However, it seems to be never published in an explicit form. I am grateful to P Exner for this information.

We see that both in this problem and in our problem (2.6)–(2.7) all the eigenvalues escape to infinity. After the subtraction of an appropriate growing term, the eigenvalues converge to those of an explicitly given operator. However, this limiting operator does not coincide with the 'most natural candidate', namely with the Neumann Laplacian on $\mathcal{L}$, or respectively on Γ. Instead, it is the Sturm–Liuville operator with the curvature-induced term, as in (5.3), or the Laplacian with other conditions at the vertices, as in (1.4).

Note that for the Neumann Laplacian in Ω_d the result is easier. Namely, the entire spectrum does not escape to infinity, so that there is no need to subtract any growing term. The nth eigenvalue of the Laplacian tends to $\lambda_n(\mathbf{H})$, where $\mathbf{H}$ is just the operator $-d^2/\mathrm{d}s^2$ along the curve. Moreover, the result extends to the case when instead of a curve one is dealing with the planar graph, see [6]. An extension to a class of non-compact graphs is also possible, see [12].

Many results of a similar nature can be found in the survey paper [5], see also [2, 4, 7].

Acknowledgments

The first result on the problem was obtained in the author's paper [9], jointly with U Smilansky. When discussing this paper, J Rubinstein conjectured that its result should extend to the general Schrödinger operators attached to the graph, instead of the harmonic oscillator as in [9]. The present paper answers his question. I use this opportunity to express my gratitude to U Smilansky and J Rubinstein and also to P Exner for a useful discussion and to the referees for the careful reading of the manuscript.

References

[1] Birman M Sh and Solomyak M 1992 Schrüodinger operator *Estimates for Number of Bound States as Function-Theoretical Problem (Am. Math. Soc. Transl. Ser. 2* vol 150*)* (Providence, RI: American Mathematical Society) pp 1–54

[2] Duclos P and Exner P 1995 Curvature-induced bound states in quantum waveguides in two and three dimensions *Rev. Math. Phys.* **7** 73–102

[3] Exner P and Ŝeba P 1989 Bound states in curved quantum waveguides *J. Math. Phys.* **30** 2574–80

[4] Froese R and Herbst I 2001 Realizing holonomic constraints in classical and quantum mechanics *Commun. Math. Phys.* **220** 489–535

[5] Kuchment P 2002 Graph models for waves in thin structures *Waves Random Media* **12** R1–R24

[6] Rubinstein J and Schatzman M 2001 Variational problems on multiply connected thin strips: I. Basic estimates and convergence of the Laplacian spectrum *Arch. Rat. Mech. Anal.* **160** 271–308

[7] Rubinstein J and Sternberg P 2003 Limits of a Ginzburg–Landau model with codimensional defects *J. Math. Phys.* **44** 1240–51

[8] Smilansky U 2004 Irreversible quantum graphs *Waves Random Media* **14** S143–S153

[9] Smilansky U and Solomyak M 2006 The quantum graph as a limit of a network of physical wires *Quantum Graphs and Their Applications (Contemporary Mathematics* vol 415*)* ed G Berkolaiko, R Carlson, S Fulling and P Kuchment (Providence, RI: American Mathematical Society) at press

[10] Solomyak M 2004 On the discrete spectrum of a family of differential operators *Funct. Anal. Appl.* **38** No. 3 70–8 (in Russian)
Solomyak M 2004 *Funct. Anal. Appl.* **38** 217–23 (English Transl.)

[11] Solomyak M 2005 On the mathematical model of the irreversible quantum graph *Algebra i Analiz* **17** No. 5 190–230 (in Russian)
Solomyak M 2006 *St.-Petersburg Math. J.* **17** (English Transl) at press

[12] Zelig D 2005 Properties of solutions of partial differetntial equations defined on human lung-shaped domains *PhD Thesis* Technion